

Hypersurfaces of Lorentzian para-Sasakian Manifolds

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Abstract. In this paper, we study the invariant and noninvariant hypersurfaces of $(1, 1, 1)$ almost contact manifolds, Lorentzian almost paracontact manifolds and Lorentzian para-Sasakian manifolds, respectively. We show that a noninvariant hypersurface of an $(1, 1, 1)$ almost contact manifold admits an almost product structure. We investigate hypersurfaces of affinely cosymplectic and normal $(1, 1, 1)$ almost contact manifolds. It is proved that a noninvariant hypersurface of a Lorentzian almost paracontact manifold is an almost product metric manifold. Some necessary and sufficient conditions have been given for a noninvariant hypersurface of a Lorentzian para-Sasakian manifold to be locally product manifold. We establish a Lorentzian para-Sasakian structure for an invariant hypersurface of a Lorentzian para-Sasakian manifold. Finally we give some examples for invariant and noninvariant hypersurfaces of a Lorentzian para-Sasakian manifold.

M.S.C. 2000. 53C25, 53C42, 53C50.

Keywords. Invariant Hypersurfaces, Non-Invariant Hypersurfaces, $(1, 1, 1)$ Almost Contact Manifolds, Lorentzian Almost Paracontact manifolds, Lorentzian para-Sasakian Manifolds.

1 Introduction

Hypersurfaces of an almost contact manifold have been studied by Blair [2], Eum [5], Goldberg and Yano [7], Ludden [8] and the others. In 1970, Goldberg and Yano [7] defined noninvariant hypersurfaces of almost contact manifolds. A hypersurface such that the transform of a tangent vector of the hypersurface by the tensor φ defining the almost contact structure is never tangent to the hypersurface is called a noninvariant hypersurface of almost contact manifold [7]. The authors [7] showed that a noninvariant hypersurface of an almost contact manifold admits an almost complex structure and a distinguished 1-form induced by the contact form of the manifold. They also investigated the noninvariant hypersurface of an almost contact metric manifold.

In 1976, Sato [13] studied a structure similar to the almost contact structure, namely almost paracontact structure. In [1], T. Adati studied hypersurfaces of an almost paracontact manifold. A. Bucki [3] considered hypersurfaces of an almost r -paracontact Riemannian manifold. Some properties of invariant hypersurfaces of an almost r -paracontact Riemannian manifold were investigated

in [4] by A. Bucki and A. Miernowski. Moreover, in [10], I. Mihai and K. Matsumoto studied submanifolds of an almost r -paracontact Riemannian manifold of P-Sasakian type. In [6] the authors studied invariant and noninvariant hypersurfaces of almost R-paracontact manifolds. Singh [14] defined (e_1, e_2, r) almost contact structure as a generalization of many known structures, which are obtained by taking particular values of (e_1, e_2) and r (see also [15]). The study of Lorentzian almost paracontact manifolds was initiated by Matsumoto in 1989 [9]. Also he introduced the notion of Lorentzian para-Sasakian (for short, LP -Sasakian) manifold. I. Mihai and R. Rosca [11] defined the same notion independently and thereafter many authors [18, 12, 16, 17] studied LP -Sasakian manifolds and their submanifolds.

In the present paper, we study invariant and noninvariant hypersurfaces of $(1, 1, 1)$ almost contact manifolds, Lorentzian almost paracontact manifolds and Lorentzian para-Sasakian manifolds, respectively. We investigate the invariant hypersurfaces with two different conditions: when the characteristic vector field ξ is everywhere tangent to the hypersurfaces and when the characteristic vector field ξ does not belong to the tangent hyperplane of the hypersurfaces. Section 2 is devoted to preliminaries. In section 3 we show that a noninvariant hypersurface of a $(1, 1, 1)$ almost contact manifold with the characteristic vector field ξ nowhere tangent to the hypersurface admits an almost product structure. In section 4 we study hypersurfaces of affinely cosymplectic and normal $(1, 1, 1)$ almost contact manifolds. In section 5 it is proved that a noninvariant hypersurface of a Lorentzian almost paracontact manifold is an almost product metric manifold. We also find a necessary and sufficient condition for a noninvariant hypersurface of a Lorentzian para-Sasakian manifold to be locally product manifold. In section 5 we establish a Lorentzian para-Sasakian structure for an invariant hypersurface of a Lorentzian para-Sasakian manifold with the characteristic vector field ξ tangent to the hypersurface. In the last section we give some examples for invariant and noninvariant hypersurfaces of an $(1, 1, 1)$ almost contact manifold, a Lorentzian almost paracontact manifold and a Lorentzian para-Sasakian manifold.

2 Preliminaries

Let \overline{M} be an n -dimensional differentiable manifold. If there exist a tensor field φ of type $(1, 1)$, r -linearly independent vector fields ξ_α and r 1-forms η^α on \overline{M} such that [15]

$$(2.1) \quad \varphi(\xi_\alpha) = 0,$$

$$(2.2) \quad \varphi^2 = e_1 I + e_2 \eta^\alpha \otimes \xi_\alpha,$$

where e_1, e_2 take values ± 1 independently, I denotes the identity map of $\Gamma(T\overline{M})$ and \otimes is the tensor product, then the structure $(\varphi, \xi_\alpha, \eta^\alpha)$ is said to be an almost (e_1, e_2) - r -contact structure or in short (e_1, e_2, r) ac structure and the manifold \overline{M} with the (e_1, e_2, r) ac structure is called an (e_1, e_2, r) ac manifold.

Let \overline{M} be an (e_1, e_2, r) ac manifold. Then the following relations hold on \overline{M} [14]:

$$(2.3) \quad \eta^\alpha \circ \varphi = 0,$$

$$(2.4) \quad \eta^\alpha(\xi_\beta) = -e_1 e_2 \delta_\beta^\alpha,$$

$$(2.5) \quad \text{rank}(\varphi) = n - r.$$

Now, consider that \overline{M} is a $(1, 1, 1)$ ac manifold. Then \overline{M} admits a Lorentzian metric \overline{g} , such that

$$(2.6) \quad \eta(\overline{X}) = \overline{g}(\overline{X}, \xi),$$

$$(2.7) \quad \overline{g}(\varphi\overline{X}, \varphi\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) + \eta(\overline{X})\eta(\overline{Y}),$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$. In this case \overline{M} is said to admit a Lorentzian almost paracontact structure $(\varphi, \xi, \eta, \overline{g})$. Then we get

$$(2.8) \quad \Phi(\overline{X}, \overline{Y}) \equiv \overline{g}(\overline{X}, \varphi\overline{Y}) \equiv \overline{g}(\varphi\overline{X}, \overline{Y}) \equiv \Phi(\overline{Y}, \overline{X}),$$

$$(2.9) \quad (\overline{\nabla}_{\overline{X}}\Phi)(\overline{Y}, \overline{Z}) = \overline{g}(\overline{Y}, (\overline{\nabla}_{\overline{X}}\varphi)\overline{Z}) = (\overline{\nabla}_{\overline{X}}\Phi)(\overline{Z}, \overline{Y}),$$

where $\overline{\nabla}$ is the Levi-Civita connection with respect to \overline{g} . It is clear that Lorentzian metric \overline{g} makes ξ a timelike unit vector field, i.e, $\overline{g}(\xi, \xi) = -1$. The manifold \overline{M} equipped with a Lorentzian almost paracontact structure $(\phi, \xi, \eta, \overline{g})$ is called a Lorentzian almost paracontact manifold (for short *LAP*-manifold) [9, 19].

A Lorentzian almost paracontact manifold \overline{M} endowed with the structure $(\varphi, \xi, \eta, \overline{g})$ is called a Lorentzian paracontact manifold (for short *LP*-manifold) [9] if

$$(2.10) \quad \Phi(\overline{X}, \overline{Y}) = \frac{1}{2}((\overline{\nabla}_{\overline{X}}\eta)\overline{Y} + (\overline{\nabla}_{\overline{Y}}\eta)\overline{X}).$$

A Lorentzian almost paracontact manifold \overline{M} endowed with the structure $(\varphi, \xi, \eta, \overline{g})$ is called a Lorentzian para Sasakian manifold (for short *LP*-Sasakian) [9] if

$$(2.11) \quad (\overline{\nabla}_{\overline{X}}\varphi)\overline{Y} = \eta(\overline{Y})\overline{X} + \overline{g}(\overline{X}, \overline{Y})\xi + 2\eta(\overline{X})\eta(\overline{Y})\xi.$$

We note that in a *LP*-Sasakian manifold the 1-form η is closed.

Let $\overline{M} \times R$ be a product manifold, where \overline{M} is an $(1, 1, 1)$ ac manifold. The tensor field J' of type $(1, 1)$ on $\overline{M} \times R$ defined by

$$(2.12) \quad J'(\overline{X}, f \frac{d}{dt}) = (\varphi\overline{X} - f\xi, \eta(\overline{X}) \frac{d}{dt}),$$

where f is a C^∞ real-valued function and $\overline{X} \in \Gamma(T\overline{M})$, satisfies $J'^2 = I$ and thus provides an almost product structure on $\overline{M} \times R$. If the induced almost product structure on $\overline{M} \times R$ is integrable then the $(1, 1, 1)$ ac structure on \overline{M} is said to be normal [15]. Since the vanishing of the Nijenhuis tensor $[J', J']$

is a necessary and sufficient condition for integrability, the condition of the normality in terms of Nijenhuis tensor $[\varphi, \varphi]$ of φ is (see [15])

$$(2.13) \quad [\varphi, \varphi] + d\eta \otimes \xi = 0,$$

where

$$(2.14) \quad [\varphi, \varphi](\overline{X}, \overline{Y}) = [\varphi\overline{X}, \varphi\overline{Y}] - \varphi[\varphi\overline{X}, \overline{Y}] - \varphi[\overline{X}, \varphi\overline{Y}] + \varphi^2[\overline{X}, \overline{Y}],$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$.

3 Non-Invariant Hypersurfaces of $(1, 1, 1)$ ac Manifolds

Let \overline{M} is a $(1, 1, 1)$ ac manifold. Consider an $(n - 1)$ -dimensional manifold M imbedded in \overline{M} with the imbedding map

$$i : M \rightarrow \overline{M},$$

and assume that for each $p \in M$ the vector $\xi_{i(p)}$ does not belong to the tangent hyperplane of the hypersurface. Then we have

$$(3.1) \quad \varphi i_* X = i_* JX + \alpha(X)\xi,$$

where J is a tensor field of type $(1, 1)$, α is a 1-form on M and i_* is the differential of the immersion i of M into \overline{M} . If $\alpha \neq 0$, then the submanifold $i(M)$ is called a noninvariant hypersurface of \overline{M} . On the other hand, if the 1-form α vanishes, $i(M)$ is called an invariant hypersurface of \overline{M} (see [7]). A hypersurface may, of course, be neither invariant nor noninvariant. Throughout this section, unless specified otherwise $i(M)$ will be a noninvariant hypersurface of the $(1, 1, 1)$ ac manifold \overline{M} .

Theorem 3.1 *If M is a noninvariant hypersurface of a $(1, 1, 1)$ ac manifold \overline{M} with ξ nowhere tangent to M , then M admits an almost product structure.*

Proof. By applying φ to (3.1) and using (2.1)-(2.4), we have

$$(3.2) \quad i_* X + \eta(i_* X)\xi = i_*(J^2 X) + \alpha(JX)\xi.$$

Then from (3.1), we get

$$J^2 X = X$$

and

$$(3.3) \quad \alpha(JX) = \eta(i_* X) = i^*(\eta X),$$

where $X \in \Gamma(TM)$ and i^* is the dual map of i_* . Thus J acts as an almost product structure on M . ■

If we define a 1-form $C\alpha$ on M by $C\alpha(X) = \alpha(JX)$ then from (3.3) we can write

$$C\alpha = i^*\eta.$$

Thus, the hypersurface M admits a 1-form α whose vanishing means that the tangent hyperplane of the hypersurface is invariant under φ .

Now, let $\bar{\nabla}$ be a symmetric affine connection on \bar{M} and define an affine connection ∇ on M with respect to the affine normal ξ by

$$(3.4) \quad \bar{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + h(X, Y)\xi,$$

where h is a symmetric tensor field of type $(0,2)$ on M which is called the second fundamental form of M with respect to ξ .

Suppose that the $(1,1,1)$ ac structure is normal. Then, the torsion field S of type $(1,2)$ on M which is defined by

$$(3.5) \quad S(\bar{X}, \bar{Y}) = [\varphi\bar{X}, \varphi\bar{Y}] - \varphi[\varphi\bar{X}, \bar{Y}] - \varphi[\bar{X}, \varphi\bar{Y}] + \varphi^2[\bar{X}, \bar{Y}] + d\eta(\bar{X}, \bar{Y})\xi,$$

for all $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$, vanishes. By taking $\bar{Y} = \xi$ in (3.5), we get

$$L_\xi\varphi = 0 \quad \text{and} \quad L_\xi\eta = 0,$$

where L_ξ is the Lie derivative operator with respect to ξ . From (3.5) the tensor field S is also expressed by

$$(3.6) \quad \begin{aligned} S(\bar{X}, \bar{Y}) &= \bar{\nabla}_{\varphi\bar{X}}(\varphi\bar{Y}) - \bar{\nabla}_{\varphi\bar{Y}}(\varphi\bar{X}) - \varphi(\bar{\nabla}_{\varphi\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}(\varphi\bar{X})) \\ &\quad - \varphi(\bar{\nabla}_{\bar{X}}(\varphi\bar{Y}) - \bar{\nabla}_{\varphi\bar{Y}}\bar{X}) + \varphi^2(\bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}) \\ &\quad + (\bar{\nabla}_{\bar{X}}\eta(\bar{Y}) - \bar{\nabla}_{\bar{Y}}\eta(\bar{X}) - \eta([\bar{X}, \bar{Y}]))\xi, \end{aligned}$$

or

$$(3.7) \quad \begin{aligned} S(\bar{X}, \bar{Y}) &= (\bar{\nabla}_{\varphi\bar{X}}\varphi)\bar{Y} - (\bar{\nabla}_{\varphi\bar{Y}}\varphi)\bar{X} + \varphi(\bar{\nabla}_{\bar{Y}}\varphi)\bar{X} - \varphi(\bar{\nabla}_{\bar{X}}\varphi)\bar{Y} \\ &\quad + [(\bar{\nabla}_{\bar{X}}\eta)\bar{Y} - (\bar{\nabla}_{\bar{Y}}\eta)\bar{X}]\xi. \end{aligned}$$

By using (3.1) and (3.4), we obtain

$$(3.8) \quad \begin{aligned} S(i_*X, i_*Y) &= i_*[J, J](X, Y) + L_\xi\varphi\{\alpha(X)i_*Y - \alpha(Y)i_*X\} \\ &\quad + \{d\alpha(JX, Y) + d\alpha(X, JY) - 2i^*\eta([X, Y])\}\xi \end{aligned}$$

Therefore, we have

Theorem 3.2 *A noninvariant hypersurface of a normal $(1,1,1)$ ac manifold \bar{M} is a locally product manifold which has a 1-form $\alpha = C^{-1}i^*\eta$ such that its differential satisfies*

$$(3.9) \quad d\alpha(JX, Y) + d\alpha(X, JY) = 2C\alpha([X, Y]).$$

Corollary 3.3 *An invariant hypersurface of a $(1,1,1)$ ac manifold is an almost product manifold. If the $(1,1,1)$ ac manifold is normal, then the almost product structure is integrable.*

Theorem 3.4 *Let ξ be an infinitesimal automorphism of the $(1, 1, 1)$ ac manifold \overline{M} . If, for every noninvariant hypersurface, the induced almost product structure J is integrable and the differential of the induced 1-form $\alpha = C^{-1}i^*\eta$ satisfies (3.9) then \overline{M} is normal.*

4 Hypersurfaces of affinely cosymplectic and normal $(1, 1, 1)$ ac manifolds

Let \overline{M} be a $(1, 1, 1)$ ac manifold with a symmetric affine connection $\overline{\nabla}$ and ∇ denotes the induced connection on the noninvariant hypersurface M . If we write

$$(4.1) \quad (\nabla_X i_*)Y = \overline{\nabla}_{i_*X} i_*Y - i_*(\nabla_X Y),$$

then the Gauss and Weingarten equations are

$$(4.2) \quad (\nabla_X i_*)Y = h(X, Y)\xi, \quad h(X, Y) = h(Y, X),$$

and

$$(4.3) \quad \overline{\nabla}_{i_*X}\xi = -i_*AX + w(X)\xi,$$

where h and A are the second fundamental tensors of type $(0, 2)$ and $(1, 1)$, respectively of M with respect to ξ , and w is a 1-form on M defining the connection on the affine normal bundle.

By using (3.1), (4.2) and (4.3) we get

$$(4.4) \quad \begin{aligned} (\overline{\nabla}_{i_*X}\varphi)i_*Y &= \overline{\nabla}_{i_*X}\varphi i_*Y - \varphi(\overline{\nabla}_{i_*X}i_*Y) \\ &= [h(X, JY) + (\nabla_X\alpha)(Y) + w(X)\alpha(Y)]\xi \\ &\quad + i_*[(\nabla_X J)Y - \alpha(Y)AX]. \end{aligned}$$

Then we will investigate the following two cases:

Case I : Let \overline{M} be an affinely cosymplectic $(1, 1, 1)$ ac manifold, that is, \overline{M} be a $(1, 1, 1)$ ac manifold with a symmetric affine connection $\overline{\nabla}$ such that

$$(4.5) \quad \overline{\nabla}\varphi = 0, \quad \overline{\nabla}\eta = 0.$$

From (3.7) we can easily see that an affinely cosymplectic $(1, 1, 1)$ ac manifold is normal. Also by using (2.1) and (2.2), we can show that (4.5) implies that

$$\overline{\nabla}\xi = 0.$$

Therefore, by (4.3), we have

$$AX = 0 \quad \text{and} \quad w(X) = 0.$$

Moreover, since $\overline{\nabla}\varphi = 0$ then from (4.4) we have

$$\nabla J = 0$$

and

$$(\nabla_X \alpha)(Y) = -h(X, JY).$$

Case II : Let \overline{M} be a normal $(1, 1, 1)$ ac manifold such that $\varphi = \overline{\nabla}\xi$. Then by using (3.1) and (4.3), we have

$$i_* JX + \alpha(X)\xi = -i_* AX + w(X)\xi$$

that is

$$J = -A$$

and

$$\alpha = w.$$

If $AX = 0$, for all $X \in \Gamma(TM)$, then from (4.3) it is obvious that $\overline{\nabla}_{i_* X}\xi$ and ξ are propotional. So affine normals are parallel along the hypersurface. In this case, the hypersurface M is said to be totally flat.

Proposition 4.1 *Let M be a noninvariant hypersurface of an affinely cosymplectic $(1, 1, 1)$ ac manifold. Then M is totally flat and*

$$\begin{aligned} \nabla J &= 0, \\ (\nabla_X \alpha)(Y) &= -h(X, JY), \\ w &= 0. \end{aligned}$$

Corollary 4.2 *Let M be an invariant hypersurface of an affinely cosymplectic $(1, 1, 1)$ ac manifold. Then*

$$\begin{aligned} \nabla J &= 0, \\ h &= 0, \\ w &= 0. \end{aligned}$$

Proposition 4.3 *Let M be a noninvariant hypersurface of a normal $(1, 1, 1)$ ac manifold such that $\varphi = \overline{\nabla}\xi$. Then*

$$J = -A$$

and

$$\alpha = w.$$

5 Hypersurfaces of Lorentzian almost paracontact manifolds

A $(1, 1, 1)$ ac manifold \overline{M} admitting a Lorentzian metric \overline{g} such that

$$(5.1) \quad \overline{g}(\overline{X}, \xi) = \eta(\overline{X})$$

$$(5.2) \quad \overline{g}(\overline{X}, \varphi \overline{Y}) \equiv \overline{g}(\varphi \overline{X}, \overline{Y}),$$

where $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$, is called Lorentzian almost paracontact manifold and denoted by $(\overline{M}, \varphi, \eta, \overline{g})$.

Proposition 5.1 *Let (M, J, α, g) be a noninvariant hypersurface of $(\overline{M}, \varphi, \eta, \overline{g})$ where g is the induced metric on M , that is, $i^*\overline{g} = g$. Then the hypersurface (M, J, α, g) admits an almost product metric*

$$(5.3) \quad G = g + \alpha \otimes \alpha.$$

Proof. From (5.2), we can write

$$(5.4) \quad \overline{g}(\varphi i_* X, i_* Y) = \overline{g}(i_* X, i_* JY).$$

By using (2.1) in (5.4), we obtain

$$(5.5) \quad \overline{g}(i_* JX, i_* Y) + \alpha(X)\eta(i_* Y) = \overline{g}(i_* X, i_* JY) + \alpha(Y)\eta(i_* X).$$

The induced metric g on (M, J, α) can be defined by

$$g(X, Y) = \overline{g}(i_* X, i_* Y).$$

So if we use (3.3) and (5.4) in (5.5), then we have

$$g(JX, Y) + \alpha(X)C\alpha(Y) = g(X, JY) + \alpha(Y)C\alpha(X),$$

that is,

$$(g + \alpha \otimes \alpha)(JX, Y) = (g + \alpha \otimes \alpha)(X, JY).$$

If we denote $g + \alpha \otimes \alpha$ by G , the proof is completed. ■

Corollary 5.2 *A noninvariant hypersurface of a Lorentzian almost paracontact manifold is an almost product metric manifold.*

Now, let define 2-forms

$$\Phi(\overline{X}, \overline{Y}) = \overline{g}(\varphi \overline{X}, \overline{Y}), \quad \overline{X}, \overline{Y} \in \Gamma(T\overline{M})$$

and

$$\Omega(X, Y) = G(JX, Y), \quad X, Y \in \Gamma(TM).$$

Φ and Ω are called the fundamental forms of the Lorentzian almost paracontact manifold $(\overline{M}, \varphi, \eta, \overline{g})$ and the submanifold (M, J, G) of \overline{M} , respectively. Then we have

Lemma 5.3 *Let Φ and Ω are the fundamental forms of $(\overline{M}, \varphi, \eta, \overline{g})$ and (M, J, α, G) , respectively. Then*

$$(5.6) \quad i^*\Phi = \Omega - C\alpha \wedge \alpha.$$

Proof. For $X, Y \in \Gamma(TM)$, by using definitions of the fundamental forms, (3.1) and (5.3), we get

$$\Phi(i_* X, i_* Y) = \Omega(X, Y) - (C\alpha \wedge \alpha)(X, Y).$$

Hence, we obtain

$$i^*\Phi(X, Y) = (\Omega - C\alpha \wedge \alpha)(X, Y).$$

■

Theorem 5.4 *Let (M, J, α, G) be a noninvariant hypersurface of the Lorentzian para-Sasakian manifold $(\overline{M}, \varphi, \eta, \overline{g})$. Then*

- (a) $J = -A$,
- (b) $\alpha = w$.

Proof. Since $(\overline{M}, \varphi, \eta, \overline{g})$ is a Lorentzian para-Sasakian manifold, we have

$$\overline{\nabla}_{i_*X}\xi = \varphi i_*X.$$

By using (4.3) and (3.1), we get

$$-i_*AX + w(X)\xi = i_*JX + \alpha(X)\xi$$

which gives

$$J = -A \quad \text{and} \quad \alpha = w.$$

■

Theorem 5.5 *If M is a noninvariant hypersurface of a Lorentzian para-Sasakian manifold $(\overline{M}, \varphi, \eta, \overline{g})$, then*

- (a) $(\nabla_X J)(Y) = \alpha(Y)JX - C\alpha(Y)X$,
- (b) $\overline{g}(i_*X, i_*Y) + 2C\alpha(X)C\alpha(Y) = h(X, JY) + (\nabla_X \alpha)(Y) + \alpha(X)\alpha(Y)$.

Proof. By using (3.1) and (4.1) we obtain

$$(5.7) \quad (\overline{\nabla}_{i_*X}\varphi)(i_*Y) = [i_*(\nabla_X J)(Y) + \alpha(Y)i_*JX] + [h(X, JY) + (\nabla_X \alpha)(Y) + \alpha(X)\alpha(Y)]\xi.$$

On the other hand, since $(\overline{M}, \varphi, \eta, \overline{g})$ is a Lorentzian para-Sasakian manifold, from (2.11) we also have

$$(5.8) \quad (\overline{\nabla}_{i_*X}\varphi)(i_*Y) = \eta(i_*Y)i_*X + \overline{g}(i_*X, i_*Y)\xi + 2\eta(i_*X)\eta(i_*Y)\xi.$$

By considering $C\alpha = i^*\eta$ and equating the components of (5.7) and (5.8), we get (a) and (b) in the theorem. ■

As an immediate consequence we have the following:

Corollary 5.6 *Let M be a noninvariant hypersurface of the Lorentzian para-Sasakian manifold $(\overline{M}, \varphi, \eta, \overline{g})$ with the induced almost product structure J . Then M is a locally product manifold if and only if*

$$(5.9) \quad \alpha(Y)JX = \alpha(JY)X.$$

Now, let \overline{M} be a (1,1,1) ac manifold and M be an invariant hypersurface of \overline{M} . Assume that for each $p \in M$ the vector $\xi_{i(p)}$ belongs to the tangent hyperplane of the hypersurface. For an invariant hypersurface of a (1,1,1) ac manifold we can write

$$(5.10) \quad \varphi i_*X = i_*\psi X,$$

where ψ is a tensor of type (1,1) on the hypersurface M and $X \in \Gamma(TM)$. Applying φ to the both sides of the equation (5.10), we get

$$(5.11) \quad i_*\psi^2 X = \varphi^2 i_* X = i_* X + \eta(i_* X)\xi.$$

If we denote

$$(5.12) \quad i_*\xi^* = \xi$$

and

$$(5.13) \quad \eta^*(X) = \eta(i_* X),$$

then we have

$$(5.14) \quad \psi^2 X = X + \eta^*(X)\xi^*.$$

Furthermore,

$$(5.15) \quad \eta^*(\psi X) = \eta(i_*\psi X) = \eta(\varphi i_* X) = 0,$$

$$(5.16) \quad \eta^*(\xi^*) = \eta(i_*\xi^*) = \eta(\xi) = -1$$

and

$$i_*\psi\xi^* = \varphi i_*\xi^* = \varphi\xi = 0,$$

that is

$$(5.17) \quad \psi\xi^* = 0.$$

Thus we have,

Theorem 5.7 *Let M be an invariant hypersurface of a $(1,1,1)$ ac manifold $(\overline{M}, \varphi, \eta, \xi)$ and $\xi \in \Gamma(TM)$. Then M is a $(1,1,1)$ ac manifold with the structure (ψ, ξ^*, η^*) where $i_*\xi^* = \xi$ and $\eta^*(X) = \eta(i_* X)$, for all $X \in \Gamma(TM)$.*

Theorem 5.8 *Let M be an invariant hypersurface of a $(1,1,1)$ ac manifold $(\overline{M}, \varphi, \eta, \xi)$ with $\xi \in \Gamma(TM)$. If \overline{M} is normal, then M is also normal.*

Proof. By using (3.5), we can write

$$(5.18) \quad \begin{aligned} S(i_* X, i_* Y) &= [\varphi, \varphi](i_* X, i_* Y) + d\eta(i_* X, i_* Y)\xi \\ &= [\varphi i_* X, \varphi i_* Y] - \varphi[\varphi i_* X, i_* Y] - \varphi[i_* X, \varphi i_* Y] \\ &\quad + \varphi^2[i_* X, i_* Y] + d\eta(i_* X, i_* Y)\xi. \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. If we use (5.10), (5.12) and (5.13) in (5.18), we get

$$\begin{aligned} S(i_* X, i_* Y) &= i_*\psi^2[X, Y] + [i_*\psi X, i_*\psi Y] - i_*\psi[X, \psi Y] - i_*\psi[\psi X, Y] \\ &\quad + \{(i_* X)(\eta^*(Y)) - (i_* Y)(\eta^*(X)) - \eta^*([X, Y])\}i_*\xi^* \\ &= i_*\{[\psi, \psi](X, Y) + d\eta^*(X, Y)\xi^*\}. \end{aligned}$$

Hence, we have the assertion of the theorem. ■

Theorem 5.9 *Let M be an invariant hypersurface of a Lorentzian almost paracontact manifold $(\overline{M}, \varphi, \eta, \overline{g})$ where $\xi \in \Gamma(TM)$. Then M is also a Lorentzian almost paracontact manifold.*

Proof. From Theorem 5.7. it follows that an invariant hypersurface M of \overline{M} is a $(1, 1, 1)$ ac manifold with the structure (ψ, ξ^*, η^*) . Let g^* be the induced metric on M . Then we have

$$(5.19) \quad g^*(\psi X, \psi Y) = \overline{g}(i_*\psi X, i_*\psi Y) = \overline{g}(\varphi i_*X, \varphi i_*Y).$$

Since \overline{M} is a Lorentzian almost paracontact manifold, then by using (5.13) in (5.19) we get

$$(5.20) \quad g^*(\psi X, \psi Y) = g^*(X, Y) + \eta^*(X)\eta^*(Y).$$

Moreover,

$$(5.21) \quad g^*(X, \xi^*) = \overline{g}(i_*X, i_*\xi^*) = \eta(i_*X) = \eta^*(X),$$

which completes the proof. ■

Theorem 5.10 *Let $(\overline{M}, \varphi, \eta, \overline{g})$ be a Lorentzian para Sasakian manifold. Then an invariant hypersurface with $\xi \in \Gamma(TM)$ of \overline{M} is also a Lorentzian para Sasakian manifold.*

Proof. Let \overline{M} be a Lorentzian para-Sasakian manifold. Then we have

$$\overline{\nabla}_{i_*X}\xi = \varphi i_*X,$$

where $\overline{\nabla}$ is a Levi-Civita connection with respect to \overline{g} . From (5.10) and (5.12), we can write

$$\overline{\nabla}_{i_*X}i_*\xi^* = i_*\psi X.$$

By using (3.4) in the last equation, we obtain

$$i_*\nabla_X\xi^* + h(X, \xi^*)N = i_*\psi X,$$

where ∇ is the induced connection on M and N is normal to M . If we consider normal and tangent components of above equation we get

$$\begin{aligned} \nabla_X\xi^* &= \psi X, \\ h(X, \xi^*) &= 0. \end{aligned}$$

Since \overline{M} be a Lorentzian para Sasakian manifold from (2.11), we have

$$(5.22) \quad (\overline{\nabla}_{i_*X}\varphi)i_*Y = \eta(i_*Y)i_*X + \overline{g}(i_*X, i_*Y)\xi + 2\eta(i_*X)\eta(i_*Y)\xi,$$

for all $X, Y \in \Gamma(TM)$. By using (5.10), (5.12) and (5.13) in (5.22), we obtain

$$(5.23) \quad (\overline{\nabla}_{i_*X}\varphi)i_*Y = i_*\{\eta^*(Y)X + \overline{g}(X, Y)\xi^* + 2\eta^*(X)\eta^*(Y)\xi^*\}.$$

On the other hand, from (3.4) and (5.10), one can get

$$\begin{aligned}
(\bar{\nabla}_{i_*X}\varphi)i_*Y &= \bar{\nabla}_{i_*X}\varphi i_*Y - \varphi(\bar{\nabla}_{i_*X}i_*Y) \\
&= \bar{\nabla}_{i_*X}i_*\psi Y - \varphi(i_*\nabla_X Y + h(X, Y)N) \\
(5.24) \quad &= i_*(\nabla_X \psi Y - \psi(\nabla_X Y)) + h(X, \psi Y)N - h(X, Y)\varphi N,
\end{aligned}$$

where ∇ is the induced connection on M and N is normal to M . By equating right hand sides of equations (5.23) and (5.24), we have

$$(\nabla_X \psi)Y = \eta^*(Y)X + \bar{g}(X, Y)\xi^* + 2\eta^*(X)\eta^*(Y)\xi^*.$$

This completes the proof. ■

6 Examples

Example 6.1 Let \bar{M} , be the 5-dimensional real number space with a coordinate system (x, y, z, t, s) . Defining

$$\begin{aligned}
\eta &= ds - dx - dz, \quad \xi = -\frac{\partial}{\partial s}, \\
\varphi\left(\frac{\partial}{\partial x}\right) &= -\frac{\partial}{\partial x} - \frac{\partial}{\partial s}, \quad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \\
\varphi\left(\frac{\partial}{\partial z}\right) &= -\frac{\partial}{\partial z} - \frac{\partial}{\partial s}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \quad \varphi\left(\frac{\partial}{\partial s}\right) = 0,
\end{aligned}$$

the set (φ, ξ, η) becomes a $(1, 1, 1)$ ac structure in \bar{M} .

Let M_1 be a hypersurface of \bar{M} which is given by $s = x$ with the imbedding map $i : M_1 \rightarrow \bar{M}$. Then

$$\{u_1 = (1, 0, 0, 0, 1), u_2 = (0, 1, 0, 0, 0), u_3 = (0, 0, 1, 0, 0), u_4 = (0, 0, 0, 1, 0)\}$$

is a local basis for the tangent hyperplane of M_1 and $N_1 = (1, 0, 0, 0, -1)$ is the normal vector field of the hypersurface. It is obvious that the characteristic vector field $\xi_{i(p)}$, $p \in M_1$, does not belong to the tangent hyperplane of M_1 . A tangent vector field of the hypersurface can be written by $X \equiv i_*X = f_1u_1 + f_2u_2 + f_3u_3 + f_4u_4$ for some smooth functions f_i , $1 \leq i \leq 4$, on M . Then we have

$$\varphi i_*X = -f_1u_1 - f_2u_2 - f_3u_3 - f_4u_4 + f_3\xi$$

which shows that M_1 is a noninvariant hypersurface of \bar{M} .

Now let us consider the hypersurface M_2 of the $(1, 1, 1)$ ac manifold \bar{M} defining by $x = y$ and let $i : M_2 \rightarrow \bar{M}$ be the imbedding map of M_2 into M . In this case the set

$$\{v_1 = (1, 1, 0, 0, 0), v_2 = (0, 0, 1, 0, 0), v_3 = (0, 0, 0, 1, 0), v_4 = (0, 0, 0, 0, 1)\}$$

is a local basis for the tangent hyperplane and $N_2 = (1, -1, 0, 0, 0)$ is the normal vector field of M_2 . The characteristic vector field belongs to the tangent

hyperplane of the hypersurface. For any tangent vector field $X \equiv i_*X = h_1v_1 + h_2v_2 + h_3v_3 + h_4v_4$ of the hypersurface we have

$$(6.1) \quad \varphi i_*X = -h_1v_1 - h_2v_2 - f_3v_3 + (h_1 + h_2)\xi$$

where h_i , $1 \leq i \leq 4$, are some smooth functions on M_2 . From (6.1) we see that that M_2 is an invariant hypersurface of \overline{M} .

Example 6.2 Let \overline{M} be the 5-dimensional real number space with a coordinate system (x, y, z, t, s) . In \overline{M} we define

$$\begin{aligned} \eta &= ds - dx, & \xi &= -\frac{\partial}{\partial s}, \\ \varphi\left(\frac{\partial}{\partial x}\right) &= \frac{\partial}{\partial x} + \frac{\partial}{\partial s}, & \varphi\left(\frac{\partial}{\partial y}\right) &= \frac{\partial}{\partial y}, \\ \varphi\left(\frac{\partial}{\partial z}\right) &= \frac{\partial}{\partial z}, & \varphi\left(\frac{\partial}{\partial t}\right) &= \frac{\partial}{\partial t}, & \varphi\left(\frac{\partial}{\partial s}\right) &= 0, \\ g &= (dx)^2 + (dy)^2 + (dz)^2 + (dt)^2 - \eta \otimes \eta. \end{aligned}$$

Then (φ, ξ, η, g) is a Lorentzian almost paracontact structure in \overline{M} .

Let M be a hypersurface of \overline{M} which is defined by $s = x$ with the imbedding map $i : M \rightarrow \overline{M}$. Then the set

$$\{u_1 = (1, 0, 0, 0, 1), u_2 = (0, 1, 0, 0, 0), u_3 = (0, 0, 1, 0, 0), u_4 = (0, 0, 0, 1, 0)\}$$

is a local basis for the tangent hyperplane of M and $N = (1, 0, 0, 0, -1)$ is the normal vector field of the hypersurface. Since $\xi_{i(p)} = \frac{1}{2}(u_1 - N)_{i(p)}$, it can be easily seen that the characteristic vector field $\xi_{i(p)}$, $p \in M$, does not belong to the tangent hyperplane of M . Moreover, since $\varphi u_1 = u_1$, $\varphi u_2 = u_2$, $\varphi u_3 = u_3$, $\varphi u_4 = u_4$, then M is an invariant hypersurface of \overline{M} with the characteristic vector field $\xi_{i(p)}$, $p \in M$, which does not belong to the tangent hyperplane of the hypersurface.

Example 6.3 Let \overline{M} be the 3-dimensional real number space with a coordinate system (x, y, z) . If we define

$$\begin{aligned} \eta &= dz, & \xi &= -\frac{\partial}{\partial z}, \\ \varphi\left(\frac{\partial}{\partial x}\right) &= -\frac{\partial}{\partial x}, & \varphi\left(\frac{\partial}{\partial y}\right) &= -\frac{\partial}{\partial y}, & \varphi\left(\frac{\partial}{\partial z}\right) &= 0, \\ g &= (dx)^2 + (dy)^2 - \eta \otimes \eta. \end{aligned}$$

on \overline{M} , then (φ, ξ, η, g) is a Lorentzian almost paracontact structure in \overline{M} .

Assume that M be a surface of \overline{M} given by $x = \arcsin y$ with the imbedding map $i : M \rightarrow \overline{M}$. Then

$$\{u_1 = (1, \sqrt{1-y^2}, 0), u_2 = (0, 0, 1)\}$$

forms a local basis for the tangent plane of M and $N = (\sqrt{1-y^2}, -1, 0)$ is the normal vector field of the surface. For any tangent vector field X of the surface we have

$$(6.2) \quad \varphi i_* X = -f_1 u_1$$

where $X \equiv i_* X = f_1 u_1 + f_2 u_2$ for some smooth functions f_1, f_2 on M . From (6.2) we obtain that M is an invariant surface of \overline{M} with the characteristic vector field $\xi_{i(p)}$, $p \in M$, belonging to the tangent plane of the surface.

Example 6.4 Let $\overline{M} = R^3$ be the 3-dimensional real number space with a co-ordinate system (x, y, z) . We define

$$(6.3) \quad \begin{aligned} \eta &= dz, \quad \xi = -\frac{\partial}{\partial z}, \\ \varphi\left(\frac{\partial}{\partial x}\right) &= \frac{\partial}{\partial x}, \quad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \\ g &= e^{-2z}(dx)^2 + e^{2z}(dy)^2 - (dz)^2. \end{aligned}$$

Then (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on \overline{M} .

Let M_1 be a surface of \overline{M} with the imbedding map $i : M_1 \rightarrow \overline{M}$ which is given by

$$z = x + y.$$

Then $u_1 = (1, 0, 1), u_2 = (0, 1, 1)$ is a local basis for the tangent plane of the surface. The vector field

$$N = (e^{2(x+y)}, e^{2(x+y)}, 1)$$

is a normal vector field of M_1 . Since

$$\xi = -\frac{1}{e^{2(x+y)} + e^{-2(x+y)} - 1}((e^{2(x+y)})u_1 + (e^{-2(x+y)})u_2 - N)$$

then for each $p \in M_1$ the characteristic vector field $\xi_{i(p)}$ does not belong to the tangent plane of the surface. A tangent vector field of the surface can be written by $X \equiv i_* X = f_1 u_1 + f_2 u_2$ for some smooth functions f_1, f_2 on M . By using (6.3) we have

$$(6.4) \quad \varphi i_* X = f_1 u_1 - f_2 u_2 + (f_1 - f_2)\xi.$$

From (3.1) and (6.4) we get

$$i_* JX = f_1 u_1 - f_2 u_2$$

and

$$\alpha(X) = f_1 - f_2$$

where J acts an almost product structure on M_1 . Consequently, M_1 is a non-invariant surface of the Lorentzian para-Sasakian manifold \overline{M} with ξ nowhere tangent to M_1 .

Let M_2 be another surface of \overline{M} which is given by

$$x = \arctan y.$$

Then $v_1 = (\frac{1}{1+y^2}, 1, 0)$, $v_2 = (0, 0, 1)$ forms a local orthogonal basis for the tangent plane of the surface and

$$N = (e^{2z}, -\frac{1}{1+y^2}e^{-2z}, 0)$$

is a normal vector field of M_2 . It is obvious that the characteristic vector field of the manifold belongs to the tangent plane of M_2 . For any tangent vector field $i_*Y \equiv Y$ of the surface where $i : M_2 \rightarrow \overline{M}$ is an imbedding map into the Lorentzian para-Sasakian manifold \overline{M} we can write $i_*Y = \gamma_1 v_1 + \gamma_2 v_2$ for some smooth functions γ_1, γ_2 on M_2 . By using (6.3) we have

$$\varphi i_*Y = -\gamma_1(v_1 - \frac{2(1+y^2)}{(1+y^2)^2 e^{2z} - e^{-2z}} N).$$

which shows that M_2 is a noninvariant surface of the Lorentzian para-Sasakian manifold \overline{M} with ξ tangent to the surface.

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